Weak convergence of Vervaat and Vervaat Error processes of long-range dependent sequences

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Abstract

Following Csörgő, Szyszkowicz and Wang (Ann. Statist. **34**, (2006), 1013–1044) we consider a long range dependent linear sequence. We prove weak convergence of the uniform Vervaat and the uniform Vervaat error processes, extending their results to distributions with unbounded support and removing normality assumption.

1 Introduction

Let $\{\epsilon_i, i \in \mathbb{Z}\}$ be a centered sequence of i.i.d. random variables with finite variance. Consider the class of stationary linear processes

$$X_i = \sum_{k=0}^{\infty} c_k \epsilon_{i-k}, \quad i \geq 1. \tag{1}$$

We assume that the sequence c_k, $k \geq 0$, is regularly varying with index $-\beta$, $\beta \in (1/2, 1)$ (written as $c_k \in RV_{-\beta}$). This means that $c_k \sim k^{-\beta} L_0(k)$ as $k \to \infty$, where L_0 is slowly varying at infinity (see e.g. [2, Sections 1.4, 1.5] for the definition of slowly varying functions). We shall refer to all such models as long range dependent (LRD) linear processes. In particular, by the Karamata Theorem, the covariances $\rho_k := \mathrm{E} X_0 X_k$ decay at the hyperbolic

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rate, $\rho_k = L(k)k^{-(2\beta-1)}$, where $\lim_{k\to\infty} L(k)/L_0^2(k) = B(2\beta-1, 1-\beta)$ and $B(\cdot,\cdot)$ is the beta-function. Consequently, since $-(2\beta-1) > -1$, the covariances are not summable.

Assume that X_1 has a continuous distribution function F and the density f, which is assumed to be positive almost everywhere. For $y \in (0,1)$ define $Q(y) = \inf\{x : F(x) \geq y\} = \inf\{x : F(x) = y\}$, the corresponding (continuous) quantile function. Given the ordered sample $X_{1:n} \leq \cdots \leq X_{n:n}$ of $X_1, \ldots, X_n$, let $F_n(x) = n^{-1}\sum_{i=1}^n 1_{\{X_i \leq x\}}$ be the empirical distribution function and $Q_n(\cdot)$ be the corresponding left-continuous sample quantile function. Define $U_i = F(X_i)$ and $E_n(x) = n^{-1}\sum_{i=1}^n 1_{\{U_i \leq x\}}$, the associated uniform empirical distribution. Denote by $U_n(\cdot)$ the corresponding uniform sample quantile function.

Let r be an integer and let

$$Y_{n,r} = \sum_{i=1}^n \sum_{1\leq j_1 < \cdots < j_r < \infty} \prod_{s=1}^r c_{j_s}\epsilon_{i-j_s}, \qquad n \geq 1,$$

so that $Y_{n,0} = n$, and $Y_{n,1} = \sum_{i=1}^n X_i$. If $1 \leq p < (2\beta-1)^{-1}$, then (cf. [11])

$$\sigma_{n,p}^2 := \mathrm{Var}(Y_{n,p}) \sim const. n^{2-p(2\beta-1)} L^{2p}(n). \tag{2}$$

In particular

$$\sigma_{n,1}^2 \sim \frac{c_\beta}{(1-\beta)(3-2\beta)} n^{3-2\beta} L^2(n) =: n^{3-2\beta} L_0^2(n).$$

Define now the general empirical, the uniform empirical, the general quantile and the uniform quantile processes respectively as follows:

$$\beta_n(x) = \sigma_{n,1}^{-1} n(F_n(x) - F(x)), \qquad x \in \mathbb{R}, \tag{3}$$

$$\alpha_n(y) = \sigma_{n,1}^{-1} n(E_n(y) - y), \qquad y \in [0,1], \tag{4}$$

$$q_n(y) = \sigma_{n,1}^{-1} n(Q(y) - Q_n(y)), \qquad y \in (0,1), \tag{5}$$

$$u_n(y) = \sigma_{n,1}^{-1} n(y - U_n(y)), \qquad y \in [0,1]. \tag{6}$$

Let

$$\tilde{R}_n(y) = \alpha_n(y) - u_n(y), \qquad y \in [0,1], \tag{7}$$

be the uniform Bahadur-Kiefer process. This process was introduced by Kiefer in [13], though not explicitly, in order to study the behavior of quantile processes via that of empirical, as initiated by Bahadur [1] for $y \in (0,1)$

fixed.

Let

$$\tilde{V}_n(t) = 2\sigma_{n,1}^{-1} n \int_0^t \tilde{R}_n(y)dy, \qquad t \in [0,1],$$

be the uniform Vervaat process and

$$\tilde{W}_n(t) = 2\sigma_{n,1}^{-1} n \int_0^t \tilde{R}_n(y)dy - \alpha_n^2(t), \qquad t \in [0,1],$$

be the uniform Vervaat error process as in [7].

Assume for a while that $\{\eta_n\}_{n\geq 1}$ is a stationary and standardized (i.e., zero-mean and unit variance) long-range dependent Gaussian sequence with a covariance structure

$$\gamma(k) := \mathrm{E}(\eta_1 \eta_{k+1}) = k^{-D}\tilde{L}(k), \qquad 0 < D < 1,$$

where $\tilde{L}$ is slowly varying at infinity. Let G be an arbitrary real-valued measurable function and define $Y_n = G(\eta_n)$, $n \geq 1$. Let F_Y be the continuous distribution function of Y_1 and $Q_Y(\cdot)$ the corresponding continuous quantile function. Define $V_i = F_Y(Y_i)$. As in Dehling and Taqqu [10], expand $1_{\{X_n \leq x\}} - F(x)$ as,

$$1_{\{Y_n \leq x\}} - F_Y(x) = \sum_{l=\tau_x}^{\infty} c_l(x) H_l(\eta_n)/l!,$$

where

$$H_l(x) = (-1)^l \exp(x^2/2) \frac{d^l}{dx^l} \exp(-x^2/2)$$

is the lth Hermite polynomial,

$$c_l(x) = \mathrm{E}\left[\left(1_{\{G(\eta_1)\leq x\}} - F_Y(x)\right) H_l(\eta_1)\right],$$

and for any $x \in \mathbb{R}$, τ_x (the Hermite rank) is the index of the first non-zero coefficient of the expansion. The uniform version is obtained as

$$1_{\{V_n \leq y\}} - y = \sum_{l=\tau_y}^{\infty} J_l(y) H_l(\eta_n)/l!,$$

where now $J_l(y) = c_l(Q_Y(y))$ for any $y \in (0,1)$.

Let $\tilde{\sigma}^2_{n,\tau} = n^{2-\tau D}\tilde{L}^\tau(n)$. Replace the constants $\sigma_{n,1}$ with $\tilde{\sigma}_{n,\tau}$ in the definitions of $\tilde{R}_n(\cdot)$, $\tilde{V}_n(\cdot)$ and $\tilde{W}_n(\cdot)$. In [7] Csörgő, Szyszkowicz and Wang (CsSzW) proved that the uniform Bahadur-Kiefer process $\tilde{R}_n(\cdot)$ converges weakly in $D([0,1])$. This phenomenon is exclusive for long range dependent sequences, since in the i.i.d. case the (uniform) Bahadur-Kiefer process cannot converge weakly. However, as it was first shown by Vervaat [16], in the i.i.d case the uniform Vervaat process does converge weakly. Obviously, in the LRD case, weak convergence of the uniform Vervaat process is implied by that of $\tilde{R}_n(\cdot)$, namely (see [7, Theorem 3.1]):

$$\tilde{V}_n(t) \Rightarrow \frac{2}{(2-\tau D)(1-\tau D)} J^2_\tau(t) Z^2_\tau, \qquad n \to \infty, \tag{8}$$

where $\Rightarrow$ denotes weak convergence in $D([0,1])$ equipped with the sup-norm, and Z_τ is a random variable defined by an appropriate integral with respect to Brownian motion (see [10]). In particular, if $\tau = 1$, then Z_1 is standard normal. Further, CsSzW [7] observed that, similarly to the i.i.d case, the limiting process associated with $\tilde{V}_n(\cdot)$ agrees with that of $\alpha^2_n(\cdot)$. Therefore, it makes sense to consider the uniform Vervaat error process $\tilde{W}_n(\cdot)$. They showed that this process converges weakly as well, via concluding

$$n\sigma^{-1}_{n,1}\tilde{W}_n(t) \Rightarrow \frac{2^{5/2}}{(2-\tau D)^{3/2}(1-\tau D)^{3/2}} J^2_\tau(t) J'_\tau(t) Z^3_\tau, \qquad n \to \infty. \tag{9}$$

This property is also exclusive for the LRD case. We refer to [3], [9], [20], [6] as well as the Introduction in [7] for motivations, probabilistic properties and applications of Bahadur-Kiefer, Vervaat and Vervaat error processes.

We note in passing that, though the results in CsSzW [7] for the uniform Bahadur-Kiefer process and, consequently, for the uniform Vervaat and Vervaat error processes, are true, their proofs are invalid, unless F_Y, the distribution of the subordinated random variable $G(\eta_1)$, is assumed to have finite support. Moreover, even then, the limiting process in (9) should be corrected via multiplying it by $\frac{1}{2}$, see [8].

In case of the Bahadur-Kiefer process, the problem of an infinite support was solved in [4] in a more general setting in the case of LRD linear sequences by using weighted approximations. However, in general, this is still not suitable for establishing the weak convergence of the Vervaat process $\tilde{V}_n(\cdot)$, unless some specific conditions are imposed on the model. The reason for the problems arising in [7], and faced up to in [4], is that, unlike in the i.i.d. case, the uniform quantile process contains information about

the quantile function associated with the random variables X_n.

Therefore, coming back to LRD linear sequences, the aim of this paper is to present an appropriate approximation result for the uniform Bahadur-Kiefer process, which will be suitable to treat the uniform Vervaat process to obtain (8), when F is assumed to have infinite support . Further, we will obtain the correct version of the weak convergence of the uniform Vervaat error process. The approach is via weighted approximation of the Bahadur-Kiefer process like in [4]. Thus, first we get the correct limiting behaviour of the Vervaat error process, second, we remove assumptions on bounded support of F, third, we remove the normality assumption on ϵ_i. This approach in fact requires very precise knowledge on the behavior of the density-quantile function $f(Q(y))$.

However, we do not extend the results in [7] in full generality, since we do not consider subordinated LRD sequences $Y_i = G(X_i)$, $i \geq 1$, where G is a measurable function. If G has a power rank 1 (see e.g. [12]), then in expense of some additional technicalities, the results will be similar as for non-subordinated case. However, if the power rank is greater than 1, the scaling factors and the limiting processes will be different.

To state our results, Let F_ϵ be the distribution function of the centered i.i.d. sequence $\{\epsilon_i, i \in \mathbb{Z}\}$ with finite 4th moment. Assume that for a given integer p, the derivatives $F_\epsilon^{(1)}, \ldots, F_\epsilon^{(p+3)}$ of F_ϵ are bounded and integrable. Note that these properties are inherited by the distribution F as well (cf. [18]). These conditions will be assumed throughout the paper with $p = 2$.

We shall need the following conditions on $fQ(\cdot) = f(Q(\cdot))$ and $f^{'}Q(\cdot) = f^{'}(Q(\cdot))$:

(A) $\sup_{y\in(0,1)} |gQ(y)|/(y(1-y))^{1-\mu} = O(1)$ for some $1/2 > \mu > 0$ and $g = f, f^{'}$;

(B) $\sup_{y\in(0,1)} |(gQ(y))^{'}|(y(1-y))^{\mu} = O(1)$ for any $\mu > 0$ and $g = f, f^{'}$. Note that $(f(Q(y)))^{'} f(Q(y)) = f^{'}(Q(y))$;

(C) $\sup_{y\in(0,1)} |(gQ(y))^{''}|(y(1-y))^{1+\mu} = O(1)$ for any $\mu > 0$ and $g = f, f^{'}$.

We shall prove the following results.

Theorem 1.1 *Assume that conditions* (A)-(C) *are fulfilled and* $\beta < 3/4$. *Then, as* $n \to \infty$,

$$\sup_{y\in[\delta_n,1-\delta_n]} \left| n\sigma_{n,1}^{-1}\tilde{R}_n(y) - \sigma_{n,1}^{-2} f'(Q(y)) \left(\sum_{i=1}^{n} X_i \right)^2 \right| = o_{\text{a.s.}}(1),$$

where $\delta_n = Cn^{-(2\beta-1)}L_0^2(n)(\log\log n)$.

Corollary 1.2 *Under the conditions of* Theorem 1.1, *as* $n \to \infty$,

$$n\sigma_{n,1}^{-1}\tilde{R}_n(y)1_{\{y\in[\delta_n,1-\delta_n]\}} \Rightarrow f'(Q(y))Z_1^2.$$

Theorem 1.3 *Under the conditions of* Theorem 1.1, *as* $n \to \infty$,

$$\tilde{V}_n(t) \Rightarrow f^2 Q(t) Z_1^2.$$

Theorem 1.4 *Under the conditions of* Theorem 1.1, *as* $n \to \infty$,

$$\sigma_{n,1}^{-1} n \tilde{W}_n(t) \Rightarrow \frac{1}{((3-2\beta)(1-\beta))^{3/2}} f^2 Q(t)(fQ)'(t)Z_1^3. \tag{10}$$

Remark 1.5 A few words on the conditions (A)-(C). Assume that $F = \Phi$ (the standard normal distribution). It follows from [15] that (A) is fulfilled. Further, $(\phi(\Phi^{-1}(y)))' = -\Phi^{-1}(y)$ is unbounded (this is actually the reason, why the proofs in [7] do not work), but (B) holds. Furthermore, $(\phi(\Phi^{-1}(y)))'' = -\frac{1}{\phi(\Phi^{-1}(y))}$, and it follows from [15] that (C) is fulfilled.

Furthermore, one can check that the conditions (A)-(B) are fulfilled for distributions with exponential or Pareto tails. To be more specific, let $f(x) = const.|x|^{-\alpha}$, $x > \delta > 0$, $\alpha > 2$. Also, in $[-\delta, \delta]$, f is interpolated smoothly to assure existence of its derivatives - note that most important issue in (A)-(C) is the tail behaviour of the density. Then, for $x > \delta$, $F(x) = 1 - cx^{-(\alpha-1)}$, $c \in (0,\infty)$, and $Q(y) = c^{1/(\alpha-1)}(1-y)^{-1/(\alpha-1)}$ for $y > \delta_0 > 1/2$. Consequently, $f(Q(y))/(1-y)^{1-\mu} = const.(1-y)^{\mu+\frac{1}{\alpha-1}}$ and $\sup_{y>\delta_0} f(Q(y))/(1-y)^{1-\mu} = O(1)$. Also, $\sup_{y>\delta_0} f'(Q(y))/(1-y)^{1-\mu} = O(1)$. The similar consideration applies to the left tail. Consequently, the condition (A) is fulfilled. Conditions (B) and (C) can be verified in a similar way.

More generally, if $f_\epsilon(x) = |x|^{-\alpha}L_1(x)$, L_1 being slowly varying at infinity, then $\lim_{x\to\infty} P(X_1 > x)/P(|\epsilon_1| > x) = const. \in (0,\infty)$ (see e.g. [14]) and by the Karamata Theorem, $\lim_{x\to\infty} f(x)/(x^{-\alpha}L_1(x)) = const. \in (0,\infty)$. Recalling that (A)-(C) are essentially the conditions on the asymptotic tail behaviour, we conclude that (A)-(C) hold.

Remark 1.6 In Theorem 1.1 we are not able to obtain the a.s. approximation on $(0,1)$. From this theorem, weak convergence of $\tilde{R}_n(y)1_{\{y\in[\delta_n,1-\delta_n]\}}$ follows, as in Corollary 1.2. We are not able to obtain weak convergence on $(0,1)$ either. However, this was not our concern in this paper. It can be done via weight functions (see [4] for more details). Nevertheless, this convergence is good enough to obtain weak convergence of both the uniform Vervaat and the uniform Vervaat error processes. The weak convergence limit in Theorem 1.4 differs from that of Proposition 3.2 in [7] by the already mentioned factor of $\frac{1}{2}$. To see this, assume that $\mathrm{E}(\epsilon_1^2)=1$ and note that parametrization of the Gaussian and the linear model yields $\tilde{L}(n)=c_\beta L^2(n)$, $D=2\beta-1$. Plugging this into (10) we see, that the result (9) should be corrected by replacing $2^{5/2}$ with $2^{3/2}$.

The problem in the proof of Proposition 3.2 in [7] comes from an inappropriate use of their Proposition 2.5.

In what follows C will denote a generic constant which may be different at each time it appears. Further, $\ell(n)$ is a slowly varying function at infinity, possibly different at each time it appears.

2 Proofs

Recall that

$$\delta_n = n^{-(2\beta-1)}L_0^2(n)(\log\log n)$$

and let

$$a_n = n^{-(\beta-1/2)}L_0(n)(\log\log n)^{1/2},$$

$$d_{n,p} = \begin{cases} n^{-(1-\beta)}L_0^{-1}(n)(\log n)^{5/2}(\log\log n)^{3/4}, & (p+1)(2\beta-1)>1 \\ n^{-p(\beta-\frac{1}{2})}L_0^{p}(n)(\log n)^{1/2}(\log\log n)^{3/4}, & (p+1)(2\beta-1)<1 \end{cases},$$

Note that $d_{n,2}=o(a_n)$ if $\beta<3/4$ and $\sigma_{n,1}^{-1}=o(d_{n,2})$.

2.1 Preliminary results

We recall the following law of the iterated logarithm for partial sums $\sum_{i=1}^n X_i$ (see, e.g., [17]):

$$\limsup_{n\to\infty}\sigma_{n,1}^{-1}(\log\log n)^{-1/2}\left|\sum_{i=1}^n X_i\right| \overset{\text{a.s.}}{=} c(\beta,1), \tag{11}$$

where where $c^2(\beta,p) = \left(\int_0^\infty x^{-\beta}(1+x)^{-\beta}dx\right)(1-\beta)^{-1}(3-2\beta)^{-1}$. Also, if $1 \le p < (2\beta-1)^{-1}$, then

$$Y_{n,p} = O_P(\sigma_{n,p}). \tag{12}$$

Lemma 2.1 *Let $p \ge 1$ be an arbitrary integer such that $p < (2\beta-1)^{-1}$. Then, as $n \to \infty$,*

$$Y_{n,p} = O_{\text{a.s.}}(\sigma_{n,p}(\log n)^{1/2}\log\log n). \tag{13}$$

Proof. Let $B_n^2 = \sigma_{n,p}^2 \log n(\log\log n)^2$. By (2), [19, Lemma 4] and Karamata's Theorem we have for $2^{d-1} < n \le 2^d$,

$$\left\|\max_{k\le n}\frac{Y_{k,p}}{B_{2^d}}\right\|_2^2 \le \frac{1}{B_{2^d}}\left(\sum_{j=0}^{d}2^{(d-j)/2}\sigma_{2^j,p}\right)^2 \le \frac{1}{B_{2^d}^2}\left(\sum_{j=0}^{d}2^{j(1-p(2\beta-1))/2}L_0^p(2^j)\right)^2$$
$$\sim \frac{1}{B_{2^d}^2}2^{2d-dp(2\beta-1)}L_0^{2p}(2^d) \sim d^{-1}(\log d)^{-2}.$$

Therefore, the result follows by the Borel-Cantelli lemma.

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The next result gives the reduction principle for the empirical processes.

Theorem 2.2 ([18]) *Let p be a positive integer. Then, as $n \to \infty$,*

$$\mathrm{E}\sup_{x\in\mathbf{R}}\left|\sum_{i=1}^{n}(1_{\{X_i\le x\}}-F(x)) + \sum_{r=1}^{p}(-1)^{r-1}F^{(r)}(x)Y_{n,r}\right|^2 = O(\Xi_n + n(\log n)^2),$$

where

$$\Xi_n = \begin{cases} O(n), & (p+1)(2\beta-1) > 1 \\ O(n^{2-(p+1)(2\beta-1)}L_0^{2(p+1)}(n)), & (p+1)(2\beta-1) < 1 \end{cases}.$$

Let $V_{n,p}(x) = \sum_{r=1}^{p}(-1)^{r-1}F^{(r)}(x)Y_{n,r}$, $x \in \mathbb{R}$ and $\tilde{V}_{n,p}(y) = V_{n,p}(Q(y))$, $y \in (0,1)$. Using Theorem 2.2 and the same argument as in the proof of Lemma 2.1, we obtain

$$\sup_{x\in\mathbf{R}}|\beta_n(x) + \sigma_{n,1}^{-1}V_{n,p}(x)| = \tag{14}$$
$$= \sigma_{n,1}^{-1}\sup_{x\in\mathbf{R}}\left|\sum_{i=1}^{n}(1_{\{X_i\le x\}}-F(x)) + V_{n,p}(x)\right| = o_{\text{a.s.}}(d_{n,p}).$$

Consequently, via $\{\alpha_n(y), y \in (0,1)\} = \{\beta_n(Q(y)), y \in (0,1)\}$,

$$\sup_{y\in(0,1)} |\alpha_n(y) + \sigma_{n,1}^{-1}\tilde{V}_{n,p}(y)| = O_{\text{a.s.}}(d_{n,p}). \tag{15}$$

We shall use this result with $p = 2$. Then, as mentioned before, $d_{n,2} = o(a_n)$ if $\beta < 3/4$.

2.2 Results on the uniform empirical and quantile processes

We have

$$\frac{\tilde{V}_{n,2}(y)}{(y(1-y))^{1/2}} = \frac{f(Q(y))}{(y(1-y))^{1/2}} \sum_{i=1}^{n} X_i - \frac{f^{'}(Q(y))}{(y(1-y))^{1/2}} Y_{n,2}.$$

Write

$$\frac{f^{'}(Q(y))}{(y(1-y))^{1/2}} = \frac{f^{'}(Q(y))}{f(Q(y))}(y(1-y))^{\mu} \frac{f(Q(y))}{(y(1-y))^{1/2+\mu}}$$

with $\mu < 1/2$. Using (A), (B) and (11) we have $\frac{\tilde{V}_{n,2}(y)}{(y(1-y))^{1/2}} = O_{\text{a.s.}}((\log\log n)^{1/2})$ uniformly on $(0,1)$.

Lemma 2.3 *Under the conditions of* Theorem 1.1, *as* $n \to \infty$,

$$\sup_{y\in[\delta_n,1-\delta_n]} \frac{|\alpha_n(y)|}{\sqrt{y(1-y)}} = O_{\text{a.s.}}((\log\log n)^{1/2}).$$

Proof. We have

$$\begin{aligned} & \sup_{y\in[\delta_n,1-\delta_n]} \frac{|\alpha_n(y)|}{\sqrt{y(1-y)}} \\ \leq & \sup_{y\in[\delta_n,1-\delta_n]} \frac{|\alpha_n(y) + \sigma_{n,1}^{-1}\tilde{V}_{n,2}(y)|}{\sqrt{y(1-y)}} + O_{\text{a.s.}}((\log\log n)^{1/2}) \\ = & \ O_{\text{a.s.}}(\delta_n^{-1/2} d_{n,2}) + O_{\text{a.s.}}((\log\log n)^{1/2}) = O_{\text{a.s.}}((\log\log n)^{1/2}), \end{aligned}$$

using (15).

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Using the method of [5, Theorem 2], we obtain the same result for the uniform quantile process.

Lemma 2.4 *Under the conditions of* Theorem 1.1*, with some* $C_0 \in (0, \infty)$*, as* $n \to \infty$*,*

$$\sup_{y \in [C_0\delta_n, 1-C_0\delta_n]} \frac{|u_n(y)|}{\sqrt{y(1-y)}} = O_{\text{a.s.}}((\log\log n)^{1/2}).$$

Next, we study the distance between the empirical and quantile processes.

Lemma 2.5 *Under the conditions of* Theorem 1.1*, as* $n \to \infty$*,*

$$\sup_{y \in (0,1)} |u_n(y) - \alpha_n(y)| = O_{\text{a.s.}}(a_n(\log\log n)^{1/2}).$$

Proof. Since

$$E_n(U_n(y)) = y + O(1/n), \tag{16}$$

we obtain from (15),

$$\begin{aligned}
&\sup_{y \in [C_0\delta_n, 1-C_0\delta_n]} |u_n(y) - \alpha_n(y)| \\
&\quad \leq \ \sigma_{n,1}^{-1} \sup_{y \in [C_0\delta_n, 1-C_0\delta_n]} |\tilde{V}_{n,2}(y) - \tilde{V}_{n,2}(U_n(y))| + O_{\text{a.s.}}(\sigma_{n,1}^{-1}) \\
&\quad \leq \ \sigma_{n,1}^{-1} \left|\sum_{i=1}^{n} X_i\right| \sup_{y \in [C_0\delta_n, 1-C_0\delta_n]} |(fQ)^{'}(\theta)||y - U_n(y)| \\
&\qquad + \sigma_{n,1}^{-1}|Y_{n,2}| \sup_{y \in [C_0\delta_n, 1-C_0\delta_n]} |(f^{'}Q)^{'}(\theta)||y - U_n(y)| + O_{\text{a.s.}}(\sigma_{n,1}^{-1}),
\end{aligned}$$

where $\theta = \theta(y, n)$ is such that $|\theta - y| \leq \sigma_{n,1}n^{-1}|u_n(y)| = O_{\text{a.s.}}((y(1-y)\sigma_{n,1}n^{-1}\log\log n)^{1/2})$ by Lemma 2.4.

Now, via Lemma 2.4,

$$\begin{aligned}
&\sup_{y \in [\delta_n, 1-\delta_n]} |(fQ)^{'}(\theta)||y - U_n(y)| \\
&\quad = \ \frac{\sigma_{n,1}}{n} \sup_{y \in [\delta_n, 1-\delta_n]} |(fQ)^{'}(\theta)||u_n(y)| \\
&\quad \leq \ \sup_{y \in [\delta_n, 1-\delta_n]} |(fQ)^{'}(\theta)|\sqrt{y(1-y)}O_{\text{a.s.}}\left(\frac{\sigma_{n,1}}{n}(\log\log n)^{1/2}\right)
\end{aligned}$$

and the bound is $O(1)O_{\text{a.s.}}\left(\frac{\sigma_{n,1}}{n}(\log\log n)^{1/2}\right)$. Indeed, by the same argument as in [5, Theorem 3],

$$\frac{y(1-y)}{\theta(1-\theta)} = O(1). \tag{17}$$

Thus, by (17) and (B),

$$\sup_{y\in[\delta_n,1-\delta_n]}|(fQ)^{'}(\theta)\sqrt{y(1-y)}|$$
$$=\sup_{y\in[\delta_n,1-\delta_n]}|(fQ)^{'}(\theta)|(\theta(1-\theta))^{1/2}\left(\frac{y(1-y)}{\theta(1-\theta)}\right)^{1/2}=O(1)$$

The second order term, in view of (A), is treated in a similar way.

Consequently, by the above calculations, (11) and (13),

$$\sup_{y\in[\delta_n,1-\delta_n]}|u_n(y)-\alpha_n(y)|=O_{\text{a.s.}}(a_n(\log\log n)^{1/2}). \tag{18}$$

Further,

$$\sup_{y\in(0,\delta_n]}|u_n(y)|=O_{\text{a.s.}}(a_n(\log\log n)^{1/2}) \tag{19}$$

by the same argument as in [5, Theorem 3]. Also,

$$\sup_{y\in(0,\delta_n]}|\alpha_n(y)|=O_{\text{a.s.}}[\delta_n^{1-\mu}\ell(n))+O_{\text{a.s.}}(d_{n,2}) \tag{20}$$

via the reduction principle, (A) and (B). Indeed,

$$\tilde{V}_{n,2}(\delta_n)=fQ(\delta_n)\sigma_{n,1}^{-1}\sum_{i=1}^{n}X_i+f^{'}(Q(\delta_n))\sigma_{n,1}^{-1}Y_{n,2}.$$

The first part is $O_{\text{a.s.}}[\delta_n^{1-\mu}\ell(n))$ by (A). For the second part, write

$$f^{'}(Q(\delta_n))\frac{\sigma_{n,2}}{\sigma_{n,1}}=\frac{f^{'}(Q(\delta_n))}{f(Q(\delta_n))}(\delta_n(1-\delta_n))^{\mu}\frac{f(Q(\delta_n))}{(\delta_n(1-\delta_n))^{\mu}}\frac{\sigma_{n,2}}{\sigma_{n,1}}=O(1)\frac{\delta_n^{1-\mu}}{\delta_n^{\mu}}\frac{\sigma_{n,2}}{\sigma_{n,1}}$$

by (A) and (B). The above bound is $O(1)$ since $\mu<1/2$. Consequently, (20) follows.

Therefore, the result of lemma follows.

$\odot$

From (13) with $p=2$, Lemma 2.5 together with the reduction principle (15) we conclude:

Corollary 2.6 *Under the conditions of* Theorem 1.1, *as* $n\to\infty$,

$$\sup_{y\in(0,1)}|u_n(y)+\sigma_{n,1}^{-1}\tilde{V}_{n,2}(y)|=O_{\text{a.s.}}(a_n(\log\log n)^{1/2}),$$

$$\sup_{y\in(0,1)}|u_n(y)+\sigma_{n,1}^{-1}f(Q(y))\sum_{i=1}^n X_i| = O_{\text{a.s.}}(a_n(\log\log n)^{1/2}(\log n)^{1/2})$$

and

$$\sup_{y\in(0,1)}|u_n(y)| = O_{\text{a.s.}}((\log\log n)^{1/2}).$$

2.3 Proof of Theorem 1.1

Let $\psi(y)=(y(1-y))^\mu$, μ from (A). Via (15) and (16) we obtain

$$\begin{aligned}
&\sup_{y\in[\delta_n,1-\delta_n]}\psi(y)\left|\tilde{R}_n(y)-\sigma_{n,1}^{-1}n^{-1}f^{'}(Q(y))\left(\sum_{i=1}^n X_i\right)^2\right| \\
\leq\ & C\sup_{y\in(0,1)}\left|\alpha_n(y)-u_n(y)+\sigma_{n,1}^{-1}(\tilde{V}_{n,2}(y)-\tilde{V}_{n,2}(U_n(y)))\right| \\
&+\sigma_{n,1}^{-1}\sup_{y\in[\delta_n,1-\delta_n]}\psi(y)\left|(\tilde{V}_{n,2}(y)-\tilde{V}_{n,2}(U_n(y)))+n^{-1}f^{'}(Q(y))\left(\sum_{i=1}^n X_i\right)^2\right| \\
=:\ & O_{\text{a.s.}}(d_{n,2})+I_2. \qquad (21)
\end{aligned}$$

Then

$$\begin{aligned}
&\sup_{y\in[\delta_n,1-\delta_n]}\psi(y)\left|(\tilde{V}_{n,2}(y)-\tilde{V}_{n,2}(U_n(y)))+n^{-1}f^{'}(Q(y))\left(\sum_{i=1}^n X_i\right)^2\right| \\
\leq\ & \sup_{y\in[\delta_n,1-\delta_n]}\psi(y)\left|\{f(Q(y))-f(Q(U_n(y)))\}\sum_{i=1}^n X_i+n^{-1}f^{'}(Q(y))\left(\sum_{i=1}^n X_i\right)^2\right| \\
&+\sup_{y\in[\delta_n,1-\delta_n]}\psi(y)\left|\left\{f^{'}(Q(U_n(y)))-f^{'}(Q(y))\right\}\right||Y_{n,2}| \\
\leq\ & \sup_{y\in[\delta_n,1-\delta_n]}\psi(y)\left|(fQ)^{'}(y)(y-U_n(y))\sum_{i=1}^n X_i+n^{-1}f^{'}(Q(y))\left(\sum_{i=1}^n X_i\right)^2\right| \\
&+\sup_{y\in[\delta_n,1-\delta_n]}\psi(y)\frac{1}{2}|(fQ)^{''}(\theta)|(y-U_n(y))^2\left|\sum_{i=1}^n X_i\right| \\
&+\sup_{y\in[\delta_n,1-\delta_n]}\psi(y)|(f^{'}Q)^{'}(\theta)||y-U_n(y)||Y_{n,2}| \\
\leq\ & \frac{\sigma_{n,1}}{n}\sup_{y\in[\delta_n,1-\delta_n]}\psi(y)\left|(fQ)^{'}(y)u_n(y)+\sigma_{n,1}^{-1}f^{'}(Q(y))\sum_{i=1}^n X_i\right|\left|\sum_{i=1}^n X_i\right| \\
&+\frac{1}{2}\sup_{y\in[\delta_n,1-\delta_n]}\psi(y)(y(1-y))|(fQ)^{''}(\theta)|\frac{(y-U_n(y))^2}{y(1-y)}\left|\sum_{i=1}^n X_i\right|+
\end{aligned}$$

$$\sup_{y\in[\delta_n,1-\delta_n]} \psi(y)(y(1-y))^{1/2}|(f^{'}Q)^{'}(\theta)|\frac{|y-U_n(y)|}{(y(1-y))^{1/2}}|Y_{n,2}|$$

with the very same θ as in Lemma 2.5.

As to the second term, by the condition (C) and (17) we have

$$\begin{aligned}
&\sup_{y\in[\delta_n,1-\delta_n]} (y(1-y))^{1+\mu}|(fQ)^{''}(\theta)| \\
&= \sup_{y\in[\delta_n,1-\delta_n]} (\theta(1-\theta))^{1+\mu}|(fQ)^{''}(\theta)|\left(\frac{y(1-y)}{\theta(1-\theta)}\right)^{1+\mu} = O(1).
\end{aligned}$$

Thus, via Lemma 2.4 and (11), the order of the second term is no greater than $O_{\text{a.s.}}(\sigma_{n,1}^2 n^{-2}\sigma_{n,1}(\log\log n)^{3/2}) = O_{\text{a.s.}}(n^{5/2-3\beta}\ell(n))$.

For the third term, via condition (A) and (17)

$$(f^{'}Q)^{'}(\theta)(y(1-y))^{1/2+\mu} = (f^{'}Q)^{'}(\theta)(\theta(1-\theta))^{1/2+\mu}\left(\frac{y(1-y)}{\theta(1-\theta)}\right)^{1/2+\mu} = O(1).$$

Consequently, the third term is $O_{\text{a.s.}}(\sigma_{n,1}n^{-1}\sigma_{n,2}\ell(n)) = O_{\text{a.s.}}(n^{5/2-3\beta}\ell(n))$.

As for the first term, we bound this by

$$\begin{aligned}
&\frac{\sigma_{n,1}}{n}\sup_{y\in[\delta_n,1-\delta_n]} \psi(y)|(fQ)^{'}(y)|\left|u_n(y)+\sigma_{n,1}^{-1}\tilde{V}_{n,2}(y)\right|\left|\sum_{i=1}^n X_i\right| \\
&\quad +n^{-1}\sup_{y\in[\delta_n,1-\delta_n]} \psi(y)\left|f^{'}(Q(y))\sum_{i=1}^n X_i-(fQ)^{'}(y)\tilde{V}_{n,2}(y)\right|\left|\sum_{i=1}^n X_i\right| =: I_3+I_4.
\end{aligned}$$

From the condition (B), (11) and Corollary 2.6, the term I_3 is

$$O_{\text{a.s.}}(\sigma_{n,1}n^{-1}a_n(\log\log n)^{1/2}\sigma_{n,1}(\log\log n)^{1/2}) = O_{\text{a.s.}}(n^{5/2-3\beta}\ell(n)).$$

Noting that $(fQ)^{'}(y)f(Q(y)) = f^{'}(Q(y))$, the term I_4 equals

$$n^{-1}\sup_{y\in[\delta_n,1-\delta_n]} \psi(y)|(fQ)^{'}(y)f^{'}(Q(y))||Y_{n,2}|\left|\sum_{i=1}^n X_i\right| = O_{\text{a.s.}}(n^{5/2-3\beta}\ell(n))$$

since $\psi(y)(fQ)^{'}(y)f^{'}(Q(y)) = O(1)$.

Thus, the term I_2 in (21) is $O_{\text{a.s.}}(\sigma_{n,1}^{-1}n^{5/2-3\beta}\ell(n))$. Consequently,

$$\begin{aligned}
&\sup_{y\in[\delta_n,1-\delta_n]} \psi(y)\left|\tilde{R}_n(y)-\sigma_{n,1}^{-1}n^{-1}f^{'}(Q(y))\left(\sum_{i=1}^n X_i\right)^2\right| \\
&= O_{\text{a.s.}}(d_{n,2})+O_{\text{a.s.}}(\sigma_{n,1}^{-1}n^{5/2-3\beta}\ell(n)) = O_{\text{a.s.}}(d_{n,2}).
\end{aligned}$$

Therefore,

$$\sup_{y\in[\delta_n,1-\delta_n]}\left|n\sigma_{n,1}^{-1}\tilde{R}_n(y)-\sigma_{n,1}^{-2}f^{'}(Q(y))\left(\sum_{i=1}^{n}X_i\right)^2\right|=O_{\text{a.s.}}(n\sigma_{n,1}^{-1}d_{n,2}\delta_n^{-\mu})=o_{\text{a.s.}}(1)$$

since $0<\mu<1/2$.

$\odot$

2.4 Proof of Theorem 1.3

We have for $t<1/2$,

$$\begin{aligned}2\sigma_{n,1}^{-1}n\int_0^t\tilde{R}_n(y)dy&=2\sigma_{n,1}^{-1}n\int_{(0,t)\cap[\delta_n,1-\delta_n]}\tilde{R}_n(y)dy\\&+O\left(\sigma_{n,1}^{-1}n\int_0^{\delta_n}|u_n(y)|dy\right)+O\left(\sigma_{n,1}^{-1}n\int_0^{\delta_n}|\alpha_n(y)|dy\right).\end{aligned}$$

The second integral is at most of the order

$$O_{\text{a.s.}}\left(\sigma_{n,1}^{-1}n\delta_n\sup_{y\in(0,\delta_n]}|u_n(y)|\right)=o_{\text{a.s.}}(1)$$

by (19). The same holds for the third one. A similar reasoning applies for $t>1/2$. Thus, the result follows from Corollary 1.2.

$\odot$

2.5 Proof of Theorem 1.4

As in [7], let

$$A_n(t)=2\sigma_{n,1}^{-1}n\int_{U_n(t)}^{t}(\alpha_n(y)-\alpha_n(t))dy.$$

Then, $\tilde{W}_n(t)=A_n(t)-\tilde{R}_n^2(t)$ (cf. (3.7) in [3]). Hence, via Theorem 1.1 and (11),

$$\sup_{t\in[\delta_n,1-\delta_n]}|A_n(t)-\tilde{W}_n(t)|=O_{\text{a.s.}}(n^{-(2\beta-1)}\ell(n)).\tag{22}$$

Via the reduction principle and the second part of Corollary 2.6,

$$\begin{aligned}\sup_{t\in(0,1)}|A_n(t)+B_n(t)|&=:\sup_{t\in(0,1)}\left|A_n(t)+2\sigma_{n,1}^{-2}n\int_{U_n(t)}^{t}(\tilde{V}_{n,2}(y)-\tilde{V}_{n,2}(t))dy\right|\qquad(23)\\&\le\ 4\sigma_{n,1}^{-1}n\sup_{y\in(0,1)}|y-U_n(y)|\sup_{y\in(0,1)}|\alpha_n(y)+\sigma_{n,1}^{-1}\tilde{V}_{n,2}(y)|=O_{\text{a.s.}}(d_{n,2}(\log\log n)^{1/2}).\end{aligned}$$

Let $C(t)=\int_0^t f(Q(y))dy$, $D(t)=\int_0^t f^{'}(Q(y))dy$. Then

$$
\begin{aligned}
B_n(t) &= 2\sigma_{n,1}^{-2}n\left(\sum_{i=1}^{n}X_i\int_{U_n(t)}^{t}(f(Q(y))-f(Q(t)))dy-Y_{n,2}\int_{U_n(t)}^{t}(f^{'}(Q(y))-f^{'}(Q(t)))dy\right)\\
&= 2\sigma_{n,1}^{-2}n\sum_{i=1}^{n}X_i\left\{C(t)-C(U_n(t))-(t-U_n(t))f(Q(t))\right\}-\\
&\quad 2\sigma_{n,1}^{-2}nY_{n,2}\left(D(t)-D(U_n(t))-(t-U_n(t))f^{'}(Q(t))\right)\\
&= 2\sigma_{n,1}^{-2}n\frac{(fQ)^{'}(t)}{2}(t-U_n(t))^2\sum_{i=1}^{n}X_i+2\sigma_{n,1}^{-2}n\frac{(fQ)^{''}(\theta)}{6}(t-U_n(t))^3\sum_{i=1}^{n}X_i-\\
&\quad 2\sigma_{n,1}^{-2}n\frac{(f^{'}Q)^{'}(t)}{2}(t-U_n(t))^2Y_{n,2}-2\sigma_{n,1}^{-2}n\frac{(f^{'}Q)^{''}(\theta)}{6}(t-U_n(t))^3Y_{n,2}.
\end{aligned}
$$

where θ is from Lemma 2.5. Consequently, by (11) and (13),

$$
\begin{aligned}
&\sup_{t\in[\delta_n,1-\delta_n]}\left|B_n(t)-\sigma_{n,1}^{-2}n\sum_{i=1}^{n}X_i(fQ)^{'}(t)(t-U_n(t))^2\right| \qquad (24)\\
&= O_{\text{a.s.}}(\sigma_{n,1}^{-1}n\ell(n))\times\sup_{t\in[\delta_n,1-\delta_n]}\frac{|t-U_n(t)|^3}{(t(1-t))^{3/2}}(fQ)^{''}(\theta)(t(1-t))^{3/2}+\\
&\quad O_{\text{a.s.}}(\sigma_{n,2}\sigma_{n,1}^{-2}n\ell(n))\times\sup_{t\in[\delta_n,1-\delta_n]}\frac{|t-U_n(t)|^2}{t(1-t)}(f^{'}Q)^{'}(t)(t(1-t))+\\
&\quad O_{\text{a.s.}}(\sigma_{n,2}\sigma_{n,1}^{-2}n\ell(n))\times\sup_{t\in[\delta_n,1-\delta_n]}\frac{|t-U_n(t)|^3}{(t(1-t))^{3/2}}(f^{'}Q)^{''}(\theta)(t(1-t))^{3/2}.
\end{aligned}
$$

Therefore, by (B), (C) and Lemma 2.4, the bound in (24) is of the order $O_{\text{a.s.}}(n^{-(2\beta-1)}\ell(n))=O_{\text{a.s.}}(d_{n,2}\ell(n))$. Consequently, via (22), (23),

$$
\sup_{t\in[\delta_n,1-\delta_n]}\left|\tilde{W}_n(t)+\sigma_{n,1}^{-2}n\sum_{i=1}^{n}X_i(fQ)^{'}(t)(t-U_n(t))^2\right|=O_{\text{a.s.}}(d_{n,2}\ell(n)).
$$

Therefore, weak convergence of $\tilde{W}_n(t)1_{\{t\in[\delta_n,1-\delta_n]\}}$ follows from Corollary 2.6 and central limit theorem for partial sums $\sum_{i=1}^{n}X_i$. Further, by (19),

$$
\sigma_{n,1}^{-2}n^2\int_0^{\delta_n}|u_n(y)|dy=O_{\text{a.s.}}(\sigma_{n,1}^{-2}n^2\delta_n)\sup_{y\in(0,\delta_n]}|u_n(y)|=o_{\text{a.s.}}(1)
$$

and the same holds if one replaces $u_n(y)$ with $\alpha_n(y)$. Thus,

$$
\sigma_{n,1}^{-2}n^2\int_0^{\delta_n}|\tilde{R}_n(y)|dy=o_{\text{a.s.}}(1).
$$

Finally, $\sigma_{n,1}^{-1} n \sup_{t\in(0,\delta_n]} \alpha_n^2(t) = O_{\rm a.s.}(\sigma_{n,1}^{-1} n d_{n,2}\ell(n)) = o_{\rm a.s.}(1)$.

$\odot$

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